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Bernstein Theorems for Elliptic Equations

THOMAS BAGBY

Department of Mathematics, Rawles Hall, Indiana University, Bloomington, Indiana 47405

AND

NORMAN LEVENBERG

Department of Mathematics and Statistics, University of Auckland, Private Bag 92019, Auckland, New Zealand

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We study solutions of the equation $L(D)\,u=0$, where $L(D)$ is an elliptic linear partial differential operator with constant coefficients and only highest order terms. For compact sets $K\subset\mathbf{R}^N$ with connected complement we prove a Bernstein theorem: if a function f on K can be extended to a solution of the equation on an open neighborhood of K, then the supnorm distance from f to the polynomial solutions of degree $\leqslant n$ decays exponentially with n. We give two proofs: a proof by duality which makes use of the theory of functions of several complex variables, and an elementary constructive proof using generalized Laurent expansions for solutions of elliptic equations. Finally, we discuss the use of orthogonal polynomial expansions, and the use of interpolation schemes, for the construction of polynomial approximations with asymptotically optimal behavior.

1. INTRODUCTION

A central theme in constructive approximation theory is the relation between the smoothness of a function and the speed at which it can be approximated by polynomials. An important result of this type is the following theorem of Walsh [W, Chap. IV], which sharpens earlier work of Bernstein concerning polynomial approximation on real intervals.

1.1. THEOREM. *Let $K\subset\mathbf{C}$ be a compact set such that $\bar{\mathbf{C}}\setminus K$ is connected and regular for the Dirichlet problem. Let f be a continuous function on K, and for each integer $n\geqslant 0$ define*

$$D_n(f,K)\equiv\inf\{\|f-p\|_K : p \text{ is a holomorphic polynomial of degree } \leqslant n\}.$$

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Let $0 \leqslant \rho < 1$. *Then* $\limsup_{n \to \infty} D_n(f, K)^{1/n} \leqslant \rho$ *if and only if* f *is the restriction to* K *of a function holomorphic on* $\{z \in \mathbf{C}: g_K(z) < \log 1/\rho\}$.

Here $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is the extended complex plane, and $\|f\|_K \equiv \sup_K |f|$; and the Green function g_K is the continuous function on $\mathbf{C}$ which vanishes on K, is harmonic on $\mathbf{C} \backslash K$, and is equal to $\log |z|$ plus a bounded function near ∞.

The purpose of the present paper is to prove a result similar to Theorem 1.1 for solutions of elliptic partial differential equations $L(D)\, u \equiv L(\partial/\partial x_1, ..., \partial/\partial x_n)\, u = 0$, where $L(x) \equiv \sum_{|\alpha| = m} a_\alpha x^\alpha$ is a nonconstant homogeneous polynomial on $\mathbf{R}^N$, with complex coefficients, which is never equal to zero on $\mathbf{R}^N \backslash \{0\}$. For each integer $n \geqslant 0$ we define the vector space $\mathscr{L}_n$ of all complex polynomials p of degree $\leqslant n$ which satisfy $L(D)\, p = 0$ on $\mathbf{R}^N$, and if f is a continuous function on a compact set $K \subset \mathbf{R}^N$ we set

$$d_n(f, K) \equiv \inf\{\|f - p\|_K : p \in \mathscr{L}_n\}.$$

Our main result is the following.

1.2. Theorem. *Let* K *be a compact subset of* $\mathbf{R}^N$ *with connected complement. Let* Ω *be an open neighborhood of* K. *Then there exists a constant* $\rho < 1$ *such that for any solution* f *of* $L(D) f = 0$ *on* Ω *we have* $\limsup_{n \to \infty} d_n(f, K)^{1/n} \leqslant \rho$.

In the case of the Cauchy–Riemann operator $L(D) = \partial/\partial\bar{z} = \frac{1}{2}(\partial/\partial x + \mathrm{i}\partial/\partial y)$ in $\mathbf{R}^2$, Theorem 1.2 yields a weak form of one direction of Theorem 1.1. Results related to Theorem 1.2 in the case of the Laplace operator $L(D) = \Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_N^2$ in $\mathbf{R}^N$ have been proved in [A] by a direct construction using spherical harmonic expansions, in [BL] by duality using techniques from several complex variables, and in [Z1, ZS] by the method of Hilbert scales. As the present paper was being completed, the authors received a preprint from Zaharjuta [Z3], who discusses many related problems for the Laplace operator, and states that most of his results can be extended to more general elliptic equations. It is possible to prove a converse to Theorem 1.2 under certain additional assumptions on the set K, and this will be discussed elsewhere by the authors (see [BL, Theorem 4.1]).

In this paper we will actually give two proofs of our main theorem: a direct proof, by constructive methods, and an indirect proof, using arguments based on duality and the Hahn–Banach theorem. The constructive proof is elementary, but requires use of generalized Laurent expansions for solutions of $L(D)\, u = 0$ near infinity. The proof by duality is shorter, but makes use of results from the theory of several complex variables.

We present some preliminary results in Section 2. In Section 3 we give the proof of our main theorem by duality. In Section 4 we review the theory of generalized Laurent expansions for solutions of $L(D)f=0$, and in Section 5 we give some estimates for the error in series expansions. In Section 6 we use series expansions and ideas from Andrievskii's work [A] to give the constructive proof of our main theorem. In the final Section 7 we discuss several techniques for the construction of polynomial approximations with asymptotically optimal behavior.

2. Notation and Preliminary Results

If $a \in \mathbf{R}^N$ and $0<r<R$, we use the notation $\mathbf{B}_r(a)=\{x \in \mathbf{R}^N : |x-a|<r\}$ and $\mathbf{A}_{r,R}(a)=\{x \in \mathbf{R}^N : r<|x-a|<R\}$, with the shortened forms $\mathbf{B}_r=\mathbf{B}_r(0)$, $\mathbf{A}_{r,R}=\mathbf{A}_{r,R}(0)$, and $\mathbf{A}_r=\mathbf{A}_{r,\infty}$. For an open set $\Omega \subset \mathbf{R}^N$, we will let $\mathscr{D}'(\Omega)$ and $\mathscr{E}'(\Omega)$ have their usual meanings from the Schwartz theory of distributions (see [H]), and we let $\langle T, \varphi \rangle$ denote the action of the distribution T on the test function φ. We let δ denote the Dirac delta measure at the origin of $\mathbf{R}^N$. If $x \in \mathbf{R}^N$ and $\alpha=(\alpha_1, ..., \alpha_N)$ is an N-tuple of nonnegative integers, the symbols x^α, $|\alpha|$, $\alpha!$ have their usual meanings and $D^\alpha=(\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_N)^{\alpha_N}$. For a polynomial $p(t) \equiv \sum a_\alpha t^\alpha$, we write $\bar{p}(t) \equiv \sum \bar{a}_\alpha t^\alpha$ and $p(D) \equiv \sum a_\alpha D^\alpha$. For each $l=0, 1, 2, ...$ we let $\mathscr{P}_l$ denote the space of all polynomials in $\mathbf{R}^N$ with complex coefficients which are homogeneous of degree l, and we regard $\mathscr{P}_l$ as a complex vector space with inner product

$$\{p, q\} \equiv p(D)\,\bar{q} = \bar{q}(D)\,p \qquad \text{for} \quad p, q \in \mathscr{P}_l. \tag{1}$$

If $|\alpha|=l$ we let $Y_\alpha(t) \equiv t^\alpha$; then the functions $\{Y_\alpha/\sqrt{a!}\}_{|\alpha|=l}$ form an orthonormal basis for $\mathscr{P}_l$.

Throughout this paper L denotes a fixed element of $\mathscr{P}_m$, $m \geqslant 1$, *which never vanishes on* $\mathbf{R}^N \setminus \{0\}$. Thus the linear partial differential operator $L(D)$ is *elliptic*. We recall that $E \in \mathscr{D}'(\mathbf{R}^N)$ is called a *fundamental solution* for $L(D)$ if $L(D)\,E=\delta$. Our work on series expansions for solutions u of $L(D)\,u=0$ depends on the existence of a particular type of fundamental solution for $L(D)$; for the following lemma see [J, Chap. 3] or [H, Chap. 7].

2.1. Lemma. *There exists a fundamental solution for $L(D)$ which is a locally integrable function on $\mathbf{R}^N$ of the form $E(x) \equiv E_1(x)+E_2(x)\log|x|$, where the restriction of E_1 to $\mathbf{R}^N \setminus \{0\}$ is real-analytic and homogeneous of degree $m-N$, and*

$$E_2 = 0 \qquad \text{if} \quad m<N \text{ or } N \text{ is odd};$$

$$E_2 \in \mathscr{P}_{m-N} \qquad \text{if} \quad m \geqslant N \text{ and } N \text{ is even}.$$

For the rest of this paper we let $E(x) \equiv E_1(x) + E_2(x) \log |x|$ *denote a fixed fundamental solution for* $L(D)$ *having the properties stated in Lemma* 2.1. We mention that the constants appearing in our paper may depend on N, L, and E. If dependence on other parameters is involved, this will be indicated. In the special case $L(D) = \Delta^s$ in $\mathbf{R}^N$, we may take

$$E(x) = \begin{cases} c_{N,s} |x|^{2s-N}, & \text{if } 2s < N \text{ or } N \text{ is odd;} \\ c_{N,s} |x|^{2s-N} \log |x|, & \text{if } 2s \geqslant N \text{ and } N \text{ is even,} \end{cases}$$

where $c_{N,s}$ is a constant.

If $x \in \mathbf{R}^N \backslash \{0\}$ is fixed, the function $y \to E(x-y)$ is a real-analytic; we may write the Taylor series expansion in y about 0,

$$E(x-y) = \sum_{l=0}^{\infty} Q_l^{(x)}(y), \tag{2}$$

where

$$Q_l^{(x)}(y) = (-1)^l \sum_{|\alpha|=l} \frac{D^\alpha E(x)}{\alpha!} y^\alpha. \tag{3}$$

This expansion is valid for y in some neighborhood of the origin in $\mathbf{R}^N$. It follows that for fixed $x \in \mathbf{R}^N \backslash \{0\}$, each polynomial $Q_l^{(x)}$ satisfies

$$L(D)\, Q_l^{(x)} \equiv 0 \qquad \text{on } \mathbf{R}^N. \tag{4}$$

In fact, we may apply the operator $L(D)$ to each monomial in (2) and combine terms to obtain the Taylor series for

$$y \to L(D)\, E(x-y), \qquad y \in \mathbf{R}^N \backslash \{x\};$$

since $L(D)\, E(x-y) \equiv 0$ near $y = 0$, each monomial in this expansion must have coefficient 0 so that $L(D)\, Q_l^{(x)} \equiv 0$ near $y = 0$. Thus, (4) follows since $Q_l^{(x)} \in \mathscr{P}_l$.

2.2. Lemma. *There exists a constant* $M_0 > 1$ *with the following property. Let* α *be a multi-index, and if* N *is even suppose that* $|\alpha| > m - N$. *Then*

$$|D^\alpha E(x)| \leqslant \alpha!\, M_0^{|\alpha|} |x|^{m-N-|\alpha|}, \qquad x \in \mathbf{R}^N \backslash \{0\}. \tag{5}$$

In particular, if l *is a nonnegative integer, and we assume that* $l > m - N$ *in case* N *is even, then*

$$|Q_l^{(x)}(y)| \leqslant |x|^{m-N-l} (M_0 |y|)^l \sum_{|\alpha|=l} 1 \qquad \textit{if} \quad x \in \mathbf{R}^N \backslash \{0\} \textit{ and } y \in \mathbf{R}^N. \tag{6}$$

Proof. Since E is real-analytic on $\mathbf{R}^N \setminus \{0\}$, it follows from the Cauchy inequalities and the Heine–Borel theorem that there exists a constant $M_0 \geqslant 1$ such that for each multi-index α we have

$$|D^\alpha E(x)| \leqslant \alpha!\, M_0^{|\alpha|} \qquad \text{if} \quad |x| = 1.$$

If N is odd or $|\alpha| > m - N$, then $D^\alpha E$ is homogeneous of degree $m - N - |\alpha|$ on $\mathbf{R}^N \setminus \{0\}$, and (5) follows. The estimate (6) follows from (3) and (5).

For the rest of this paper, M_0 denotes the constant of Lemma 2.2.

2.3. COROLLARY. *Let $M > M_0$, and $r > 0$. Then the series $\sum_{l=0}^{\infty} |Q_l^{(x)}(y)|$ converges uniformly, and Eq. (2) holds, for $|x| \geqslant Mr$ and $|y| \leqslant r$.*

Proof. If $A = \{(x, y) \in \mathbf{R}^N \times \mathbf{R}^N : |x| \geqslant Mr \text{ and } |y| \leqslant r\}$, and $l_1 = \max\{0, m - N + 1\}$, then from (3) and Lemma 2.2 we see that

$$\begin{aligned}
\sum_{l=l_1}^{\infty} \sup_{(x,y) \in A} |Q_l^{(x)}(y)| &\leqslant \sum_{l=l_1}^{\infty} (Mr)^{m-N-l} (M_0 r)^l \sum_{|\alpha| = l} 1 \\
&\leqslant (Mr)^{m-N} \sum_{|\alpha| \geqslant 0} \left(\frac{M_0}{M}\right)^{|\alpha|} \\
&= (Mr)^{m-N} \left(1 - \frac{M_0}{M}\right)^{-N}.
\end{aligned}$$

It now follows from the Weierstrass test that the series $\sum_{l=0}^{\infty} |Q_l^{(x)}(y)|$ converges uniformly for $(x, y) \in A$.

Now fix $\rho > r$ with $M_0 \rho < Mr$, and fix $x \in \mathbf{R}^N$ with $|x| \geqslant Mr$. From the result in the preceding paragraph and (4) we see that $F(y) \equiv \sum_{l=0}^{\infty} Q_l^{(x)}(y)$ is a well-defined, continuous function on $\mathbf{B}_\rho$ which satisfies $L(D) F = 0$ in the sense of distributions; from the ellipticity of $L(D)$ we conclude that F is real-analytic on $\mathbf{B}_\rho$. Since the functions $F(y)$ and $G(y) \equiv E(x - y)$ are real-analytic on $\mathbf{B}_\rho$ and agree near $y = 0$, they must agree on $\mathbf{B}_\rho$. This shows that Eq. (2) holds for $y \in \bar{\mathbf{B}}_r \subset \mathbf{B}_\rho$, which completes the proof of Corollary 2.3.

3. PROOF OF THE MAIN THEOREM BY DUALITY

In this section, we given the proof of Theorem 1.2 by means of duality. This proof may be compared with the proof for the case of harmonic functions in [BL, Theorem 3.1]; there the authors make essential use of the Kelvin transform for harmonic functions, which is not available in the present generality. We use instead an argument based on the following "two-constants" lemma for holomorphic functions on subsets of $\mathbf{C}^N$.

3.1. LEMMA. *Let $\tilde{U}$ be a bounded domain in $\mathbf{C}^N$. Let $A \subset \tilde{U} \cap \mathbf{R}^N$ be a nonempty open subset of $\mathbf{R}^N$, and H a compact subset of $\tilde{U}$. Then there exists a constant $a = a(\tilde{U}, A, H) \in (0, 1]$ with the following property. If g is a holomorphic function on $\tilde{U}$ which satisfies $|g| \leqslant M$ on $\tilde{U}$ and $|g| \leqslant m < M$ on A, then*

$$|g| \leqslant m^a M^{1-a} \qquad \text{on } H.$$

Lemma 3.1 follows from a more general two-constants lemma [Kl, Proposition 4.5.6] for *plurisubharmonic* functions; we recall that a real-valued function u defined on a domain $\tilde{U} \subset \mathbf{C}^N$ is plurisubharmonic on $\tilde{U}$ if u is uppersemicontinuous in $\tilde{U}$ and, for every complex line $l \subset \mathbf{C}^N$, the restriction of u to (components of) $\tilde{U} \cap l$ is subharmonic. One applies the general two-constants lemma to the plurisubharmonic function $u \equiv \log |f|$. We mention that in our setting, the inequality $a > 0$ follows from the fact that A is *not pluripolar*; that is, the only plurisubharmonic function u on $\mathbf{C}^N$ which is equal to $-\infty$ on A is $u \equiv -\infty$ (see [BL, Remark 4.2(b); Kl, Proposition 4.5.4]).

We remark that for $N \geqslant 2$, any set $A \subset \mathbf{R}^N \subset \mathbf{C}^N = \mathbf{R}^{2N}$ is *polar* as a subset of $\mathbf{R}^{2N}$; that is, there exists a subharmonic function $u \not\equiv -\infty$ on $\mathbf{R}^{2N}$ which is equal to $-\infty$ on A. It is therefore essential for us to use techniques from the theory of functions of several complex variables.

We now prove Theorem 1.2. We may assume that the set Ω in the theorem is bounded, so for the proof we fix a compact set $K \subset \mathbf{R}^N$ such that $\mathbf{R}^N \backslash K$ is connected, and a bounded open set $\Omega \supset K$. We fix a function $\psi \in C_0^\infty(\Omega)$ which is identically equal to one on an open neighborhood D of K. The connected manifold $\mathbf{R}^N \cup \{\infty\} \backslash K$ has an exhaustion by an increasing sequence of relatively compact subregions, and one of these subregions must contain the compact set $\mathbf{R}^N \cup \{\infty\} \backslash D$; we let G denote this subregion, with the point at infinity removed. We let $R_0 = \sup_{y \in K} |y|$, and fix a number $R_1 > R_0$ such that $\Omega \subset \mathbf{B}_{M_0 R_1}$.

Now suppose that f is a solution of $L(D) f = 0$ on Ω. For each fixed integer $n \geqslant 0$, it follows from the Hahn–Banach theorem and the Riesz representation theorem that there is a complex Borel measure $\mu = \mu_n$ of total variation one supported on K such that

$$\int p \, d\mu = 0 \qquad \text{for all} \quad p \in \mathscr{L}_n,$$

and

$$\int f \, d\mu = d_n.$$

We may regard $F \equiv \psi f \in C_0^\infty(\Omega) \subset C_0^\infty(\mathbf{R}^N)$, and then

$$d_n = \int_K F\, d\mu = \mu * \check{F}(0),$$

where $\check{F}(x) \equiv F(-x)$. We have $\mu * \check{F} = (\mu * \check{F}) * \delta = (\mu * \check{F}) * L(D)\, E = \tilde{\mu} * L(D)\, \check{F}$, where $\tilde{\mu}(x) \equiv (\mu * E)(x)$, and hence

$$d_n = (-1)^m \int_{\Omega \setminus D} \tilde{\mu}(x)\, L(D)\, F(x)\, dx. \tag{7}$$

From now on we will assume that $n \geqslant m - N$. Then for $|x| \geqslant 2M_0 R_1$ we obtain the estimate

$$\begin{aligned}
|\tilde{\mu}(x)| &= \left| \int_K \sum_{l=n+1}^{\infty} Q_l^{(x)}(y)\, d\mu(y) \right| \leqslant \sum_{l=n+1}^{\infty} \sup_{y \in K} |Q_l^{(x)}(y)| \\
&\leqslant \sum_{l=n+1}^{\infty} |x|^{m-N-l} \sup_{y \in K} \{(M_0\, |y|)^l\} \sum_{|\alpha|=l} 1 \\
&\leqslant \sum_{l=n+1}^{\infty} (2M_0 R_1)^{m-N-l} (M_0 R_0)^l \sum_{|\alpha|=l} 1 \\
&= (2M_0 R_1)^{m-N} \sum_{l=n+1}^{\infty} \left(\frac{R_0}{R_1}\right)^l \left(\frac{1}{2}\right)^l \sum_{|\alpha|=l} 1 \\
&\leqslant (2M_0 R_1)^{m-N} \sum_{l=n+1}^{\infty} \left(\frac{R_0}{R_1}\right)^{n+1} \left(\frac{1}{2}\right)^l \sum_{|\alpha|=l} 1 \\
&\leqslant (2M_0 R_1)^{m-N} \left(\frac{R_0}{R_1}\right)^{n+1} \sum_{|\alpha| \geqslant 0} \left(\frac{1}{2}\right)^{|\alpha|} \\
&= c \left(\frac{R_0}{R_1}\right)^{n+1},
\end{aligned} \tag{8}$$

where $c = 2^m (M_0 R_1)^{m-N}$; in this string the first equality follows from Corollary 2.3; the first inequality follows from the fact that μ has total variation equal to one; the second inequality follows from Lemma 2.2; and the third inequality follows from the fact that each index l is nonnegative and satisfies $m - N - l < m - N - n \leqslant 0$. We will actually apply estimate (8) for those points x in the open annulus $A = \mathbf{A}_{2M_0R_1,\, 3M_0R_1}$.

Using the fact that G is arcwise-connected, it is easy to see that $G \cap \bar{\mathbf{B}}_{3M_0R_1}$ is also; it follows that $H \equiv \bar{G} \cap \bar{\mathbf{B}}_{3M_0R_1}$ is a connected, compact set which contains $\Omega \setminus D$ and the annulus A.

We note that for each $R>0$ there exist positive numbers $r=r(R)<R$ and $C=C(R)$ with the following property: if u is any solution of $L(D)\,u=0$ on a ball $B=\mathbf{B}_R(x)\subset\mathbf{R}^N$, then there is a holomorphic function $\tilde{u}$ on the complex ball $\tilde{B}=\tilde{\mathbf{B}}_r(x)\equiv\{z\in\mathbf{C}^N: |z-x|<r\}$ which agrees with u on the real ball $\mathbf{B}_r(x)$ and satisfies $\|\tilde{u}\|_{\tilde{B}}\leqslant C\,\|u\|_B$ (see [ABG, Lemma 2]). Now for each $x\in H$ we may find a radius $R(x)$ such that the closure of the ball $B\equiv\mathbf{B}_{R(x)}(x)$ is disjoint from K, and by the preceding remark there is a holomorphic function g_x on $\tilde{B}\equiv\tilde{\mathbf{B}}_{r(R(x))}(x)$ which agrees with $\tilde{\mu}$ on the real ball $\mathbf{B}_{r(R(x))}(x)$ and satisfies

$$\|g_x\|_{\tilde{B}}\leqslant C(R(x))\;\|\tilde{\mu}\|_B.$$

By compactness of H, we can find a finite number of points $x_1, ..., x_M\in H$ such that the corresponding smaller real balls $\{\mathbf{B}_{r(R(x_i))}(x_i)\}$ cover H. Thus

$$\tilde{U}\equiv\bigcup\tilde{\mathbf{B}}_{r(R(x_i))}(x_i)$$

is a bounded domain in $\mathbf{C}^N$ which contains H; and we obtain a well-defined holomorphic function g on $\tilde{U}$ which agrees with $\tilde{\mu}$ on the intersection $\tilde{U}\cap\mathbf{R}^N$, by setting $g=g_{x_i}$ on $\tilde{\mathbf{B}}_{r(R(x_i))}(x_i)$. If we now set $U=\bigcup\mathbf{B}_{R(x_i)}(x_i)$, then since μ has total variation equal to one, we have the finite bound

$$|\tilde{\mu}|\leqslant Q\equiv\sup_{x\in U,\,y\in K}|E(x-y)|\qquad\text{on } U.$$

It follows that

$$\|g\|_{\tilde{U}}\leqslant\tilde{C}\;\|\tilde{\mu}\|_U\leqslant\tilde{C}Q,\tag{9}$$

where $\tilde{C}=\max C(R(x_i))$ is independent of n. Now if $a=a(\tilde{U}, A, H)$ denotes the constant of Lemma 3.1, then in view of estimates (8) and (9) we may apply 3.1 to the holomorphic function g on $\tilde{U}$ to obtain

$$|\tilde{\mu}(x)|=|g(x)|=\leqslant(\tilde{C}Q)^{1-a}\left(c\left(\frac{R_0}{R_1}\right)^{n+1}\right)^a,\qquad x\in H.\tag{10}$$

From (7) and (10) we see that

$$d_n\leqslant(\tilde{C}Q)^{1-a}\,c^a\left(\frac{R_0}{R_1}\right)^{(n+1)a}(\sup_{\Omega\backslash D}|L(D)\,F|)\,\lambda(\Omega\backslash D),$$

where λ denotes Lebesgue measure. Since $\tilde{C}$, Q, c, R_0, R_1, and $a>0$ are independent of n, this competes the proof of Theorem 2.1.

4. Generalized Laurent Expansions for Solutions of Elliptic Equations

All solutions of the homogeneous equation $L(D)\,u=0$ are real-analytic [H, Theorem 4.4.3], so that we can expand these solutions locally in power series. In this paper we will also need generalized Laurent expansions for solutions of the homogeneous equation $L(D)\,u=0$ on a neighborhood of infinity (that is, on the complement of a compact set). Theorems concerning series expansions for solutions of elliptic equations were obtained by Lopatinsky, John, Wachman, Balch [Bal], Žemukov [Ž], Harvey and Polking [HP], Meshkov, and Tarkhanov [T]. We give here a sharpened version of the Laurent expansion in [Ž]; a more general Laurent expansion, for solutions of elliptic systems, is developed in [T]. If $L(D)$ is the Cauchy–Riemann operator $\partial/\partial\bar{z}$ on $\mathbf{R}^2$, these series expansions give special cases of the classical Laurent expansion. If $L(D)$ is the Laplace operator Δ, these series expansions give special cases of the well-known expansion of harmonic functions in terms of spherical harmonics (see [K, Chap. 5]).

We now state the Laurent expansion theorem. We define $\mathscr{G}\equiv\{w\in C^\infty(\mathbf{R}^N):\ L(D)\,w=0 \text{ in } \mathbf{R}^N\}$. For each l we define $\mathscr{H}_l\equiv\{h\in\mathscr{P}_l : \bar{L}(D)\,h=0 \text{ in } \mathbf{R}^N\}$; we recall that $\mathscr{P}_l$ carries the inner product (1), and let π_l be the orthogonal projection from $\mathscr{P}_l$ onto the subspace $\mathscr{H}_l$.

4.1. Theorem. *There exists a constant $M>1$ with the following property. If $a\in\mathbf{R}^N$, $r>0$, and u is a solution of $L(D)\,u=0$ on $\mathbf{A}_r(a)$, then there exists a unique sequence $w\in\mathscr{G}$, $h_0\in\mathscr{H}_0$, $h_1\in\mathscr{H}_1$, ... such that*

$$u(x)=w(x)+\sum_{l=0}^{\infty} h_l(D)\,E(x-a) \tag{11}$$

uniformly on $\mathbf{A}_{Mr}(a)$.

The series (11) is called a *generalized Laurent expansion for u about a.* If $a\in\mathbf{R}^N$, we have the generalized Laurent expansion

$$u(x)\equiv w(x)+\sum_{l=0}^{\infty} h_l(D)\,E(x) \tag{12}$$

for a function u about the origin if and only if $u(x-a)\equiv w(x-a)+\sum_{l=0}^{\infty} h_l(D)\,E(x-a)$ is the generalized Laurent expansion for the function $u(\cdot-a)$ about the point a; for this reason we will often restrict our attention to generalized Laurent expansions about the origin. We now give explicit formulas for the terms in the expansion.

4.2. THEOREM. *Let u be a solution of $L(D)\,u=0$ on a neighborhood of infinity, with generalized Laurent expansion* (12) *about the origin. Suppose that $\tilde{u}\in\mathscr{D}'(\mathbf{R}^N)$ coincides with u on a neighborhood of infinity. Then*

$$w=\tilde{u}-E*L(D)\,\tilde{u},$$

and for each l we have

$$h_l(D)\,E(x)=\langle L(D)\,\tilde{u},Q_l^{(x)}\rangle,\qquad x\in\mathbf{R}^N\backslash\{0\}.$$

4.3. *Remark.* If $u\in\mathscr{D}'(\mathbf{R}^N)$, and u is a solution of $L(D)\,u=0$ on a neighborhood of infinity with Laurent expansion (12) about the origin, then the following conditions are equivalent:

(a) $w=0$.

(b) $u=E*L(D)\,u$.

(c) $u=E*T$ for some distribution $T\in\mathscr{E}'(\mathbf{R}^N)$.

In fact, the equivalence of (a) and (b) follows from Theorem 4.2 with $\tilde{u}=u$. Clearly (b) implies (c); and if (c) holds, we have

$$E*L(D)\,u=E*L(D)(E*T)=E*(\delta*T)=E*T=u,$$

so (b) holds.

If u is a solution of $L(D)\,u=0$ on a neighborhood of infinity which satisfies the equivalent conditions in 4.3, then the generalized Laurent expansion of u about any point $a\in\mathbf{R}^N$ will be of the form $u(x)\equiv\sum_{l=0}^{\infty}h_l(D)\,E(x-a)$, and we say that u has *zero entire part.*

Before giving the proof of Theorems 4.1 and 4.2 we state a well-known lemma [F, St, V].

4.4. LEMMA. *If $l\in\{0,1,...,m-1\}$ we have $\mathscr{H}_l=\mathscr{P}_l$. If $l\geqslant m$, then $\mathscr{H}_l$ is the orthogonal complement of $\{Lp\colon p\in\mathscr{P}_{l-m}\}$ in $\mathscr{P}_l$.*

Proof. The first assertion is obvious, and the second follows from the fact that for $p\in\mathscr{P}_{l-m}$ and $q\in\mathscr{P}_l$ we have

$$\{Lp,q\}=L(D)\,p(D)\,\bar{q}=p(D)\,L(D)\,\bar{q}=\{p,L(D)\,\bar{q}\}.$$

We now prove Theorems 4.1 and 4.2. We will prove that Theorem 4.1 holds with any constant M which is greater than the constant M_0 of Lemma 2.2. It suffices to prove Theorem 4.1 for $a=0$, and in this case the theorem follows from (1°) and (2°)(a) below. Theorem 4.2 follows from (2°)(b) and (2°)(c).

(1°) *Let* $M > M_0$. *Let* u *satisfy* $L(D)\,u = 0$ *on* $\mathbf{A}_r$, *where* $r > 0$. *Then there exists a sequence* $w \in \mathscr{G}$, $h_0 \in \mathscr{H}_0$, $h_1 \in \mathscr{H}_1, \ldots$ *such that* (12) *holds uniformly for* $|x| \geqslant Mr$.

Proof of (1°). Choose $\rho > r$ with $M_0\,\rho < Mr$. It is easy to construct a function $u_0 \in C^\infty(\mathbf{R}^N)$ which coincides with u on $\mathbf{A}_{(r+\rho)/2}$. We define $w = u_0 - E * L(D)\,u_0$ and note that

$$L(D)\,w = L(D)\,u_0 - (L(D)\,E) * L(D)\,u_0 = L(D)\,u_0 - \delta * L(D)\,u_0 = 0,$$

so $w \in \mathscr{G}$. If $|x| \geqslant Mr$, we conclude from Corollary 2.3 that

$$\begin{aligned} u(x) - w(x) = [E * L(D)\,u_0](x) &= \int E(x-y)\,L(D)\,u_0(y)\,dy \\ &= \sum_{l=0}^{\infty} p_l(D)\,E(x), \end{aligned} \tag{13}$$

where $p_l \equiv (-1)^l \sum_{|\alpha|=l} (\langle L(D)\,u_0, Y_\alpha\rangle/\alpha!)\,Y_\alpha \in \mathscr{P}_l$; the series in (13) converges uniformly for $|x| \geqslant M_0 r$. If we let $h_l \equiv \pi_l\,p_l$ then from Lemma 4.4 we see that $p_l(D)\,E(x) = h_l(D)\,E(x)$ for $x \neq 0$, so (1°) follows from (13).

(2°) *Let* u *be a solution of* $L(D)\,u = 0$ *on* $\mathbf{A}_r$, *where* $r \geqslant 0$. *Let* $\tilde{r} \geqslant r$, *and suppose* $\tilde{u} \in \mathscr{D}'(\mathbf{R}^N)$ *coincides with* u *on* $\mathbf{A}_{\tilde{r}}$. *Let* $\rho \geqslant r$ *and suppose* $w \in \mathscr{G}$, $h_0 \in \mathscr{H}_0$, $h_1 \in \mathscr{H}_1, \ldots$ *have the property that* (12) *holds uniformly for* $|x| \geqslant \rho$. *Then*

(a) $\{h_l, q\} = (-1)^l \langle L(D)\,\tilde{u}, \bar{q}\rangle$ *if* $q \in \mathscr{H}_l$,

(b) $h_l(D)\,E(x) = (-1)^l \sum_{|\alpha|=l} (\langle L(D)\,\tilde{u}, Y_\alpha\rangle/\alpha!)\,D^\alpha E(x)$,

(c) $w = \tilde{u} - E * L(D)\,\tilde{u}$.

Proof of (2°). (a) Let $\phi \in C_0^\infty(\mathbf{R}^N)$ be identically equal to one on a neighborhood of $\mathbf{R}^N \backslash (\mathbf{A}_{\tilde{r}} \cap \mathbf{A}_\rho)$. If $q \in \mathscr{H}_l$, then $L(D)(\bar{q}\phi)$ is supported on a compact subset of $\mathbf{A}_{\tilde{r}} \cap \mathbf{A}_\rho$. Thus

$$\begin{aligned} \langle L(D)\,\tilde{u}, \bar{q}\rangle &= \langle L(D)[\tilde{u} - w], \bar{q}\phi\rangle \\ &= (-1)^m \langle \tilde{u} - w, L(D)(\bar{q}\phi)\rangle \\ &= (-1)^m \langle u - w, L(D)(\bar{q}\phi)\rangle \\ &= (-1)^m \sum_{k=0}^{\infty} \langle h_k(D)\,E, L(D)(\bar{q}\phi)\rangle \\ &= \sum_{k=0}^{\infty} \langle (-1)^k L(D)\,E, h_k(D)(\bar{q}\phi)\rangle \\ &= \sum_{k=0}^{\infty} (-1)^k \langle \delta, h_k(D)(\bar{q}\phi)\rangle \\ &= (-1)^l \{h_l, q\}. \end{aligned}$$

(b) Define $p_l \equiv (-1)^l \sum_{|\alpha|=l} (\langle L(D)\,\tilde{u}, Y_\alpha \rangle/\alpha!)\, Y_\alpha \in \mathscr{P}_l$. If $q \in \mathscr{H}_l$, it follows from (a) that $\{h_l, q\} = (-1)^l \langle L(D)\,\tilde{u}, \bar{q} \rangle = (-1)^l \langle L(D)\,\tilde{u}, \sum_{|\alpha|=l} (\{Y_\alpha, q\}\, Y_\alpha/\alpha!) \rangle = \{p_l, q\}$. It follows that $h_l = \pi_l p_l$. From Lemma 4.4 we conclude that $h_l(D)\, E(x) = p_l(D)\, E(x)$ for $x \neq 0$, which proves (b).

(c) From the proof of (1°) we see that there are functions $u_0 \in C^\infty(\mathbf{R}^N)$, $v \in \mathscr{G}$, $g_0 \in \mathscr{H}_0$, $g_1 \in \mathscr{H}_1, \ldots$ such that u_0 coincides with u on a neighborhood of infinity, $v = u_0 - E * L(D)\, u_0$, and $u(x) = v(x) + \sum_{l=0}^\infty g_l(D)\, E(x)$ uniformly on a neighborhood of infinity. From (a) it is clear that $h_l = g_l$ for each l. We conclude that $w(x) = v(x)$ for $|x|$ large, and hence for all $x \in \mathbf{R}^N$ by the real-analyticity of elements of $\mathscr{G}$. Since the distribution $T \equiv u_0 - \tilde{u}$ lies in $\mathscr{E}'(\mathbf{R}^N)$, we have

$$E * L(D)\, u_0 - E * L(D)\, \tilde{u} = E * L(D)\, T - (L(D)\, E) * T = \delta * T = T = u_0 - \tilde{u}$$

and hence $w = v = u_0 - E * L(D)\, u_0 = \tilde{u} - E * L(D)\, \tilde{u}$.

5. Error Estimates for Series Expansions

Solutions of the homogeneous equation $L(D)\, u = 0$ can be represented locally by power series, and near infinity by generalized Laurent expansions. In the present section we estimate the truncation error for these series expansions.

We first discuss the truncation error for local series expansions. Let u be a solution of $L(D)\, u = 0$ on $\mathbf{B}_r$, where $r > 0$. For all x in an open neighborhood U of the origin the Maclaurin expansion of u must converge uniformly to u; if we group together the terms of the same order we see that for $x \in U$ we have

$$u(x) = \sum_{l=0}^{\infty} q_l(x), \tag{14}$$

where

$$q_l(x) \equiv \sum_{|\alpha|=l} \frac{D^\alpha u(0)}{\alpha!} x^\alpha. \tag{15}$$

The argument used to prove (4) shows that each polynomial q_l satisfies $L(D)\, q_l \equiv 0$ on $\mathbf{R}^N$.

5.1. Theorem. *There exists a constant $c \in (0, 1)$ with the following property. Let u be a solution of $L(D)\, u = 0$ on $\mathbf{B}_r$, where $r > 0$, with expansion* (14), (15). *Let l_0 be a positive integer, and $s(x) \equiv \sum_{0 \leq l < l_0} q_l(x)$. Then*

$$|u(x) - s(x)| \leq \left(\frac{|x|}{cr}\right)^{l_0} \sup_{\mathbf{B}_r} |u|, \qquad |x| \leq cr.$$

Proof. It follows from [ABG, Lemma 2] that there exist constants $\rho \in (0, N^{-1/2})$ and $C_0 > 1$ with the following property: if $r > 0$ and u is a solution of $L(D)\,u = 0$ on $\mathbf{B}_r$, then there is a holomorphic function $\tilde{u}$ on the polydisk $\Pi_{\rho r} \equiv \{(z_1, ..., z_N) \in \mathbf{C}^N : \text{each } |z_j| < \rho r\}$ with $\tilde{u} = u$ on $\mathbf{R}^N \cap \Pi_{\rho r}$ and

$$\sup_{\Pi_{\rho r}} |\tilde{u}(z)| \leqslant C_0 \sup_{\mathbf{B}_r} |u(x)|. \tag{16}$$

Now under the hypotheses of the theorem we have for $|x| \leqslant \rho r/2$ the estimate

$$\begin{aligned}
|u(x) - s(x)| &\leqslant \sum_{l=l_0}^{\infty} |q_l(x)| \\
&\leqslant C_0 (\sup_{\mathbf{B}_r} |u|) \sum_{l=l_0}^{\infty} \left(\frac{2\,|x|}{\rho r}\right)^l \left(\frac{1}{2}\right)^l \sum_{|\alpha| = l} 1 \\
&\leqslant C_0 (\sup_{\mathbf{B}_r} |u|) \sum_{l=l_0}^{\infty} \left(\frac{2\,|x|}{\rho r}\right)^{l_0} \left(\frac{1}{2}\right)^l \sum_{|\alpha| = l} 1 \\
&\leqslant C_0 (\sup_{\mathbf{B}_r} |u|) \left(\frac{2\,|x|}{\rho r}\right)^{l_0} \sum_{|\alpha| \geqslant 0} \left(\frac{1}{2}\right)^{|\alpha|} \\
&= 2^N C_0 (\sup_{\mathbf{B}_r} |u|) \left(\frac{2\,|x|}{\rho r}\right)^{l_0},
\end{aligned}$$

where the second inequality follows from applying the Cauchy inequalities to $\tilde{u}$ on $\Pi_{\rho r}$ and using the estimate (16). Thus Theorem 5.1 holds with $c = \rho/(2^{N+1} C_0)$.

Our next result is an estimate for the truncation error in the generalized Laurent expansion (12).

5.2. THEOREM. *There is a constant $J > 1$ with the following property. Let u be a solution of $L(D)\,u = 0$ on $\mathbf{A}_r$, where $r > 0$, with generalized Laurent expansion $\sum_{l=0}^{\infty} h_l(D)\, E(x)$. If l_0 is a positive integer satisfying $l_0 > m - N$, and $s(x) = \sum_{0 \leqslant l < l_0} h_l(D)\, E(x)$, then*

$$|u(x) - s(x)| \leqslant \left(\frac{|x|}{r}\right)^{m-N} \left(\frac{Jr}{|x|}\right)^{l_0} \sup_{\mathbf{A}_{r,2r}} |u|, \qquad |x| \geqslant Jr.$$

The proof of Theorem 5.2 will depend on the following well-known theorem. It may be proved by the argument used in [B, Lemma 3.5].

5.3. THEOREM. *Let K be a compact subset of the open set $\Omega \subset \mathbf{R}^N$. For each multi-index α, there exists a constant $C = C(K, \Omega, \alpha)$ such that if u satisfies $L(D)\,u = 0$ on Ω, then*

$$\sup_K |D^\alpha u| \leqslant C \sup_\Omega |u|.$$

5.4. COROLLARY. *For each multi-index α there exists a constant $C = C(\alpha)$ with the following property. If u satifies $L(D)\,u = 0$ on $\mathbf{A}_{r,2r}$, where $r > 0$, then*

$$\sup_{\bar{\mathbf{A}}_{4r/3,\,5r/3}} |D^\alpha u| \leqslant C r^{-|\alpha|} \sup_{\mathbf{A}_{r,2r}} |u|.$$

Proof. This follows from applying Theorem 5.3 to the function $v(y) \equiv u(ry)$, with $K = \bar{\mathbf{A}}_{4/3,\,5/3}$ and $\Omega = \mathbf{A}_{1,2}$.

Proof of Theorem 5.2. We recall that M_0 denotes the constant of Lemma 2.2, and that 2.3 and 4.1 hold with any constant $M > M_0$. Let $\eta \in C^\infty(\mathbf{R}^N)$ be a fixed function such that $\eta \equiv 0$ on $\bar{\mathbf{B}}_{4/3}$ and $\eta \equiv 1$ on $\mathbf{R}^N \backslash \mathbf{B}_{5/3}$. If we define

$$N_l = \sup\{|D^\alpha \eta(x)| : x \in \mathbf{R}^N,\ |\alpha| = l\} \tag{17}$$

and $\eta_r(x) \equiv \eta(x/r)$, then

$$|D^\alpha \eta_r| \leqslant N_{|\alpha|} r^{-|\alpha|} \qquad \text{on } \mathbf{R}^N. \tag{18}$$

We define the function $\tilde{u} \in C^\infty(\mathbf{R}^N)$ to be equal to $\eta_r u$ on $\mathbf{A}_r$ and 0 elsewhere. Then $\tilde{u}$ coincides with u on a neighborhood of infinity, so by use of 2.3, 4.1, and 4.2 we obtain

$$u(x) - s(x) = \int_{4r/3 \leqslant |y| \leqslant 5r/3} \left[\sum_{l=l_0}^{\infty} Q_l^{(x)}(y)\right] L(D)\,\tilde{u}(y)\,dy, \qquad |x| \geqslant 2M_0 r. \tag{19}$$

To estimate the sum in the bracket, we note that for $|x| \geqslant 2M_0 r$ and $|y| \leqslant 5r/3$ we have from Lemma 2.2

$$\begin{aligned}
\sum_{l=l_0}^{\infty} |Q_l^{(x)}(y)| &\leqslant \sum_{l=l_0}^{\infty} |x|^{m-N} \left(\frac{2M_0 r}{|x|}\right)^l \left(\frac{5}{6}\right)^l \sum_{|\alpha|=l} 1 \\
&\leqslant \sum_{l=l_0}^{\infty} |x|^{m-N} \left(\frac{2M_0 r}{|x|}\right)^{l_0} \left(\frac{5}{6}\right)^l \sum_{|\alpha|=l} 1 \\
&\leqslant (2M_0 r)^{l_0} |x|^{m-N-l_0} \sum_{|\alpha| \geqslant 0} \left(\frac{5}{6}\right)^{|\alpha|} \\
&= 6^N (2M_0 r)^{l_0} |x|^{m-N-l_0}.
\end{aligned} \tag{20}$$

We now note that there exist complex numbers $c_{\alpha\beta}$, depending only on $L(D)$, such that

$$L(D)\,\tilde{u}(y) = \sum_{|\alpha+\beta|=m} c_{\alpha\beta}(D^{\alpha}\eta_r(y))(D^{\beta}u(y)). \tag{21}$$

Combining Corollary 5.4, (18), (19), (20), and (21), we obtain

$$|u(x)-s(x)| \leqslant C_0\left(\frac{|x|}{r}\right)^{m-N}\left(\frac{2M_0 r}{|x|}\right)^{l_0} \sup_{\mathbf{A}_{r,2r}} |u|, \qquad |x| \geqslant 2M_0 r, \tag{22}$$

where $C_0 = 12^N V_N \sum_{|\alpha+\beta|=m} |c_{\alpha\beta}|\, N_{|\alpha|}\, C(\beta)$; here V_N denotes the volume of the unit ball in $\mathbf{R}^N$, and $C(\beta)$ denotes the constant of Corollary 5.4. From (22) we obtain Theorem 5.2 with $J = (\max\{C_0, 2M_0\})^2$.

6. Proof of the Main Theorem by Constructive Techniques

In this section we give the constructive proof of Theorem 1.2 in case $m<N$ or N is odd. This argument may be extended to give a constructive proof of Theorem 1.2 in case $m \geqslant N$ and N is even, starting with a version of Lemma 2.2, for all multi-indices α, which takes into account the logarithmic terms in the fundamental solution E; we omit the details.

6.1. Lemma. *Assume that $m<N$, or that N is odd. If $A>1$ and $T>1$, then there is a constant $C=C(A,T)$ with the following property. Let $r>0$ and $z \in \mathbf{B}_r$. Let u be a solution of $L(D)\,u=0$ on $\mathbf{A}_r$ with generalized Laurent expansion $\sum_{l=0}^{\infty} h_l(D)\,E(x)$. If l_0 is a positive integer satisfying $l_0 > m-N$, and $s(x) = \sum_{0 \leqslant l < l_0} h_l(D)\,E(x)$, then*

$$\sup_{\mathbf{A}_{Ar,\,TAr}(z)} |s| \leqslant C^{l_0} \sup_{\mathbf{A}_{r,2r}} |u|.$$

Proof. The proof depends on the elementary inclusion

$$\mathbf{A}_{Ar,\,TAr}(z) \subset \mathbf{A}_{(A-1)r,\,(AT+1)r}(0). \tag{23}$$

As in the proof of Theorem 5.2, we let $\eta \in C^{\infty}(\mathbf{R}^N)$ be a fixed function such that $\eta = 0$ on $\bar{\mathbf{B}}_{4/3}$ and $\eta = 1$ on $\mathbf{R}^N \backslash \mathbf{B}_{5/3}$. If we define N_l by (17), and $\eta_r(x) \equiv \eta(x/r)$, then (18) holds. We define the function $\tilde{u} \in C^{\infty}(\mathbf{R}^N)$ to be equal to $\eta_r u$ on $\mathbf{A}_r$ and 0 elsewhere. Then $\tilde{u}$ coincides with u on a neighborhood of infinity, so from Theorem 4.2 we obtain

$$s(x) = \int_{4r/3 \leqslant |y| \leqslant 5r/3} \left[\sum_{0 \leqslant l < l_0} Q_l^{(x)}(y)\right] L(D)\,\tilde{u}(y)\,dy, \qquad x \in \mathbf{R}^N \backslash \{0\}. \tag{24}$$

To estimate the sum in the bracket, we note that for $|x| \geqslant (A-1)\,r$ and $|y| \leqslant 5r/3$ we have by Lemma 2.2

$$\begin{aligned}
\sum_{0 \leqslant l < l_0} |Q_l^{(x)}(y)| &\leqslant \sum_{0 \leqslant l < l_0} |x|^{m-N} \left(\frac{2M_0 r}{(A-1)\,r}\right)^l \left(\frac{5}{6}\right)^l \sum_{|\alpha| = l} 1 \\
&\leqslant \sum_{0 \leqslant l < l_0} |x|^{m-N}\, C_1^{l_0-1} \left(\frac{5}{6}\right)^l \sum_{|\alpha| = l} 1 \\
&\leqslant |x|^{m-N}\, C_1^{l_0-1} \sum_{|\alpha| \geqslant 0} \left(\frac{5}{6}\right)^{|\alpha|} \\
&= 6^N\, |x|^{m-N} C_1^{l_0-1}, \qquad (25)
\end{aligned}$$

where M_0 is the constant of Lemma 2.2, and $C_1 = \max\{1, 2M_0/(A-1)\}$.

We next recall that there exist complex numbers $c_{\alpha\beta}$, depending only on $L(D)$, such that (21) holds. Combining Corollary 5.4, (18), (24), (25), and (21), we see that

$$|s(x)| \leqslant C_2 C_1^{l_0-1} \left(\frac{|x|}{r}\right)^{m-N} \sup_{\mathbf{A}_{r,2r}} |u|, \qquad |x| \geqslant (A-1)\,r, \qquad (26)$$

where $C_2 = 12^N V_N \sum_{|\alpha+\beta| = m} |c_{\alpha\beta}|\, N_{|\alpha|}\, C(\beta)$; here V_N denotes the volume of the unit ball in $\mathbf{R}^N$, and $C(\beta)$ denotes the constant of Corollary 5.4. From (23) and (26) we obtain Lemma 6.1 with $C = \max\{C_1, C_2(A-1)^{m-N}, C_2(AT+1)^{m-N}\}$.

We now prove Theorem 1.2 under the assumption that $m < N$ or N is odd. We let $c \in (0, 1)$, $J > 1$, and $C(A, T) > 1$ denote the constants of 5.1, 5.2, and 6.1, respectively; A and T will be chosen below. We may write $K \subset \bar{\mathbf{B}}_{R_0}$, where $R_0 > 0$, and we set $R = 2R_0/c$, so that

$$|x| \leqslant \frac{cR}{2} \qquad \text{if} \quad x \in K. \qquad (27)$$

The connected manifold $\mathbf{R}^N \cup \{\infty\} \backslash K$ has an exhaustion by an increasing sequence of relatively compact subregions, and one of these subregions must contain the compact set $\mathbf{R}^N \cup \{\infty\} \backslash \Omega$; we let G denote this subregion, with the point at infinity removed. Let $\chi \in C_0^\infty(\Omega)$ be identically equal to one on an open neighborhood of $\Omega \backslash G$. We regard $\tilde{f} \equiv \chi f \in C_0^\infty(\Omega) \subset C_0^\infty(\mathbf{R}^N)$, and then

$$\tilde{f}(x) = \int E(x-y)\, L(D)\, \tilde{f}(y)\, dy, \qquad x \in \mathbf{R}^N.$$

Using the fact that G is arcwise-connected, it is easy to see that $G \cap \bar{\mathbf{B}}_{R+2}$ is also; it follows that $H \equiv \bar{G} \cap \bar{\mathbf{B}}_{R+2}$ is a connected, compact set. We let d be the distance between the disjoint compact sets K and H, and define $r = \min\{1, d/(3J+2)\}$. The family of all open balls of radius r which intersect H is an open cover of H; by the Heine–Borel theorem there is a finite subfamily $\mathscr{B}$ which covers H. Note that the distance from the center of any ball $B \in \mathscr{B}$ to K is at least $(3J+1)\, r$.

We may select nonnegative functions $\varphi_B \in C_0^\infty(B)$ for $B \in \mathscr{B}$, such that $\sum_{B \in \mathscr{B}} \varphi_B \equiv 1$ on H, and hence on supp $L(D)\tilde{f}$. If we define

$$f_B(x) \equiv \int_B E(x-y)\, \varphi_B(y)\, L(D)\, \tilde{f}(y)\, dy,$$

then $f = \tilde{f} = \sum_{B \in \mathscr{B}} f_B$ on K. Theorem 1.2 now follows from the following claim: *for each* $\tilde{B} \in \mathscr{B}$ *there is a constant* $\rho = \rho(\tilde{B}) \in (0, 1)$ *with the following property: for every solution* u *of* $L(D)\, u = 0$ *on* $\mathbf{R}^N \backslash (\text{supp}\ \varphi_{\tilde{B}})$ *with zero entire part we have* $\limsup_{n \to \infty} d_n(u, K)^{1/n} \leqslant \rho$. For the rest of this section we fix the ball $\tilde{B} \in \mathscr{B}$, and prove this claim. The proof requires four preliminary results.

(1°) *The set* $\bigcup_{B \in \mathscr{B}} B$ *is connected.*

In fact, this set may be regarded as the union of the connected set H and the balls $B \in \mathscr{B}$, and each of these balls intersects H.

(2°) *There is a finite sequence of open balls of radius* r,

$$\tilde{B} = \mathbf{B}_r(a_0),\ \mathbf{B}_r(a_1),\ \ldots,\ \mathbf{B}_r(a_Z),$$

such that $\bar{\mathbf{B}}_r(a_Z) \cap \bar{\mathbf{B}}_R = \varnothing$; *each center* $a_i \in \mathbf{B}_r(a_{i-1})$ (*for* $1 \leqslant i \leqslant Z$); *and the distance from each center* a_i *to* K *is at least* $3Jr$.

In fact, using (1°) and an elementary connectedness argument, we see that there is an *even* integer Z and a sequence $\tilde{B} = \mathbf{B}_r(a_0)$, $\mathbf{B}_r(a_2)$, $\mathbf{B}_r(a_4), \ldots, \mathbf{B}_r(a_Z)$ in $\mathscr{B}$ such that $\bar{\mathbf{B}}_r(a_Z) \cap \bar{\mathbf{B}}_R = \varnothing$, and each ball $\mathbf{B}_r(a_{2i})$ intersects $\mathbf{B}_r(a_{2i+2})$. If we define a_{2i+1} to be the midpoint of the line segment joining a_{2i} and a_{2i+2}, then (2°) holds.

The open balls $\mathbf{B}_r(a_i)$ selected in (2°) will be fixed for the rest of the proof of 1.2. The constants appearing in this proof are understood to depend on the compact set K and on the balls selected in (2°).

(3°) *There is a constant* $A > 1$ *such that the distance from each center* a_i *to* K *is at least* $2JA^i r$, *and* $\bar{\mathbf{B}}_{A^Z r}(a_Z) \cap \bar{\mathbf{B}}_R = \varnothing$. *Moreover, there is a constant* $T = T(A) > 1$ *such that* $\bar{\mathbf{B}}_R \subset \bar{\mathbf{B}}_{TA^Z r}(a_Z)$.

(This is clear from the properties in (2°).)

(4°) *There are positive constants* ε_0, C, *and* v *with the following property: For every solution* u *of* $L(D)\,u=0$ *on* $\mathbf{R}^N\backslash(\operatorname{supp}\varphi_B)$ *which has zero entire part and satisfies*

$$\sup_{\mathbf{A}_{r,2r}(a_0)} |u| = 1, \tag{28}$$

and every $\varepsilon \in (0, \varepsilon_0)$, *there exists a solution* v *of* $L(D)\,v=0$ *on* $\mathbf{B}_R$ *such that*

$$\|u-v\|_K \leqslant \varepsilon$$

and

$$\|v\|_{\mathbf{B}_R} \leqslant \frac{C}{\varepsilon^{\nu}}.$$

To prove (4°), we define

$$C_0 = \max\left\{\left(\frac{|x-a_i|}{A^i r}\right)^{m-N} : x \in K, 0 \leqslant i \leqslant Z\right\},$$

and we let $\varepsilon_0 = C_0 \cdot \min\{1, 2^{N-m}\}$. Now let u be a solution of $L(D)\,u=0$ on $\mathbf{R}^N\backslash(\operatorname{supp}\varphi_B)$ satisfying (28), and $\varepsilon \in (0, \varepsilon_0)$. We may then select a positive integer $n > m-N$ such that

$$2^{n-1} \leqslant \frac{C_0 Z}{\varepsilon} < 2^n. \tag{29}$$

We let $t \geqslant 2$ be an integer such that $2^{t-1} > \max\{C(A, 2), C(A, T)\}$.

We now construct the desired solution v of $L(D)\,v=0$ on $\mathbf{B}_R$ by adapting a technique used by Andrievskii in his study of harmonic functions [A]. We set $s_0 = u$, and we inductively define s_{i+1} to be the first nt^i terms in the generalized Laurent expansion of s_i about a_i (for $0 \leqslant i \leqslant Z-1$). (That is, we inductively define $s_{i+1}(x) = \sum_{0 \leqslant l < nt^i} h_l(D)\,E(x-a_i)$, where the generalized Laurent expansion of s_i about a_i is $s_i(x) = \sum_{l=0}^{\infty} h_l(D)\,E(x-a_i)$.) We then define $v = s_Z$.

From (3°), 5.2, and 6.1 we conclude that

$$\|s_{i+1} - s_i\|_K \leqslant \frac{C_0}{2^{nt^i}} \|s_i\|_{\mathbf{A}_{A^i r, 2A^i r}(a_i)}, \qquad 0 \leqslant i \leqslant Z-2, \tag{30}$$

and

$$\|s_{i+1}\|_{\mathbf{A}_{A^{i+1}r, 2A^{i+1}r}(a_{i+1})} \leqslant C(A, 2)^{nt^i} \|s_i\|_{\mathbf{A}_{A^i r, 2A^i r}(a_i)}, \qquad 0 \leqslant i \leqslant Z-2. \tag{31}$$

From (31) we obtain

$$\|s_i\|_{\mathbf{A}_{A^i r,\, 2A^i r}(a_i)} \leqslant C(A, 2)^{n(1+t+t^2+\cdots+t^{i-1})} = C(A, 2)^{n((t^i-1)/(t-1))} \leqslant 2^{n(t^i-1)},$$
$$1 \leqslant i \leqslant Z-1. \tag{32}$$

From (3°), 5.2, and 6.1 we conclude that

$$\|s_Z - s_{Z-1}\|_K \leqslant \frac{C_0}{2^{t^{Z-1}n}} \|s_{Z-1}\|_{\mathbf{A}_{A^{Z-1}r,\, 2A^{Z-1}r}(a_{Z-1})} \tag{33}$$

and

$$\|s_Z\|_{\mathbf{A}_{A^Z r,\, TA^Z r}(a_Z)} \leqslant C(A, T)^{nt^{Z-1}} \|s_{Z-1}\|_{\mathbf{A}_{A^{Z-1}r,\, 2A^{Z-1}r}(a_{Z-1})}. \tag{34}$$

From (34) and (32) we obtain

$$\|s_Z\|_{\mathbf{A}_{A^Z r,\, TA^Z r}(a_Z)} \leqslant C(A, T)^{nt^{Z-1}} \cdot 2^{n(t^{Z-1}-1)} \leqslant 2^{n(t^Z-1)}. \tag{35}$$

We now have

$$\|s_{i+1} - s_i\|_K \leqslant \frac{C_0}{2^{t^i n}} 2^{n(t^i-1)} = \frac{C_0}{2^n} \leqslant \frac{\varepsilon}{Z}, \qquad 0 \leqslant i \leqslant Z-1, \tag{36}$$

where the first inequality follows from (30) and (32) for $0 \leqslant i \leqslant Z-2$, and from (33) and (32) for $i = Z-1$; and the last inequality follows from (29). We conclude from (36) that

$$\|s_Z - u\|_K \leqslant \varepsilon. \tag{37}$$

From (3°), (29), (35), and (37) we see that (4°) holds with $v = s_Z$, the constants of (4°) being $C = (2C_0Z)^{t^Z-1}$ and $\nu = t^Z - 1$.

We now complete the proof of the claim above. We continue to let ε_0, C, and ν denote the constants of (4°), and we fix a number $\delta \in (0, 1)$ such that $\delta^\nu > 3/4$. It suffices to prove the claim for any solution u of $L(D)\,u = 0$ on $\mathbf{R}^N \backslash (\operatorname{supp} \varphi_B)$ satisfying (28), and we let u be a fixed function with these properties. For each positive integer l such that $\delta^l < \varepsilon_0$ we may apply (4°), with $\varepsilon = \delta^l$, to obtain a solution v_l of $L(D)\,v_l = 0$ on $\mathbf{B}_R$ such that

$$\|u - v_l\|_K \leqslant \delta^l \tag{38}$$

and

$$\sup_{\mathbf{B}_R} |v_l| \leqslant \frac{C}{\delta^{l\nu}}; \tag{39}$$

and according to 5.1 and (27) there is a polynomial $s_l \in \mathscr{L}_{l-1} \subset \mathscr{L}_l$ such that

$$\|v_l - s_l\|_K \leqslant \frac{1}{2^l} \sup_{\mathbf{B}_R} |v_l|. \tag{40}$$

From (38), (39), and (40) we conclude that $\|u - s_l\|_K \leqslant \delta^l + C(2/3)^l$. The claim above follows from this, and we have completed the constructive proof of Theorem 1.2 in case $m < N$ or N is odd.

7. Construction of Polynomial Approximations by Orthogonal Expansions or by Interpolation

In this section we give conditions under which one may use orthogonal polynomial expansions or interpolating polynomials to obtain approximations to a function on a compact set $K \subset \mathbf{R}^N$ with asymptotically optimal behavior.

If μ is a positive measure with support in the compact set $K \subset \mathbf{C}^N$, we say that the pair (K, μ) has the *Bernstein–Markov property* provided that for each $\varepsilon > 0$ there exists $M > 0$ such that $\|p\|_K \leqslant M(1+\varepsilon)^{\deg p} \|p\|_2$, for all holomorphic polynomials p; here $\|p\|^2 \equiv \int_K |p|^2 \, d\mu$. By combining results of Nguyen Thanh Van and Zeriahi [NZ] and Plesniak [P] we obtain the following result: if $K \subset \mathbf{R}^N \subset \mathbf{C}^N$ has the property that every point of ∂K is analytically accessible from int K, and μ_N is N-dimensional Lebesgue measure restricted to K, then the pair (K, μ_N) has the Bernstein–Markov property. Here we use the notation ∂K and int K for the boundary and the interior of K in $\mathbf{R}^N$; and we say that a point $a \in \partial K$ is *analytically accessible* from int K provided that there exists a real-analytic function $h: (-1, 1) \to \mathbf{R}^N$ such that $h((0, 1)) \subset \operatorname{int} K$ and $h(0) = a$.

The following theorem shows that if (K, μ) has the Bernstein–Markov property, then orthogonal polynomial expansions may be used to construct polynomial approximations with asymptotically optimal behavior.

7.1. Theorem. *Let $K \subset \mathbf{R}^N$ be compact, and let μ be a measure on K such that the pair (K, μ) has the Bernstein–Markov property. If f is a continuous function on K such that* $\limsup_{n \to \infty} d_n(f, K)^{1/n} = \rho < 1$, *and $\{p_n\}$ is the sequence of best $L^2(\mu)$-approximants to f in $\mathscr{L}_n$, then* $\limsup_{n \to \infty} \|f - p_n\|_K^{1/n} = \rho$.

Proof. Without loss of generality we may assume that $\mu(K) = 1$. Let $r \in (\rho, 1)$ be arbitrary. Then there exist a constant $M > 0$ and polynomials

$q_n \in \mathscr{L}_n$ such that $\|f-q_n\|_K \leqslant Mr^n$ for $n=1, 2, \ldots$. For each n we then obtain

$$\|p_n - f\|_2 \leqslant \|q_n - f\|_2 \leqslant \|q_n - f\|_K \leqslant Mr^n.$$

Hence $\|p_n - p_{n-1}\|_2 \leqslant Mr^n(1+r^{-1})$ and, in particular, $p_0 + \sum_{n=1}^{\infty} (p_n - p_{n-1})$ converges to f in $L^2(\mu)$. The polynomial $p_n(x) \equiv \sum_{|\alpha| \leqslant n} a_\alpha^{(n)} x^\alpha$ extends to a holomorphic polynomial $P_n(z) \equiv \sum_{|\alpha| \leqslant n} a_\alpha^{(n)} z^\alpha$ of degree $\leqslant n$ in $\mathbf{C}^N$; we then see from the Bernstein–Markov property that for each $\varepsilon \in (0, r^{-1}-1)$ there is a constant $\tilde{M} > 0$ such that

$$\|p_n - p_{n-1}\|_K \leqslant \tilde{M}(1+\varepsilon)^n \|p_n - p_{n-1}\|_2 \leqslant \tilde{M}M(1+r^{-1})[(1+\varepsilon)r]^n.$$

Thus, $p_0 + \sum_{n=1}^{\infty} (p_n - p_{n-1})$ converges uniformly to f on K and

$$\begin{aligned}\|f - p_n\|_K &= \left\| \sum_{n+1}^{\infty} (p_k - p_{k-1}) \right\|_K \\ &\leqslant \tilde{M}M(1+r^{-1})[(1+\varepsilon)r]^{n+1} [1-(1+\varepsilon)r]^{-1},\end{aligned}$$

so that $\limsup_{n\to\infty} \|f-p_n\|_K^{1/n} \leqslant (1+\varepsilon)r$. The theorem follows.

Finally, we discuss the construction of asymptotically optimal polynomials by interpolation. We may order a basis $e_1 = e_1(x), e_2, e_3, \ldots$ for $\bigcup_n \mathscr{L}_n$ by increasing degree, with any ordering for those polynomials of the same degree: if $m(n) = \dim \mathscr{L}_n$, the set $\{e_1, \ldots, e_{m(n)}\}$ forms a basis for $\mathscr{L}_n$. Given a compact set $K \subset \mathbf{R}^N$, choose $m(n)$ points $A^{(n)} \equiv \{A_1^{(n)}, \ldots, A_{m(n)}^{(n)}\} \subset K$ and form the generalized *Vandermonde determinant*.

$$V_n(A^{(n)}) \equiv \det[e_i(A_j^{(n)})]_{i,j=1,\ldots,m(n)}.$$

If $V_n(A^{(n)}) \neq 0$, we can form the fundamental interpolating polynomials

$$l_j^{(n)}(x) \equiv \frac{V_n(A_1^{(n)}, \ldots, x, \ldots, A_{m(n)}^{(n)})}{V_n(A^{(n)})}, \qquad j = 1, \ldots, m(n).$$

(Here the x in the right-hand side occurs in the jth slot.) Note that $l_j^{(n)}(A_i^{(n)}) = \delta_{ij}$ and each $l_j^{(n)}$ is a linear combination of $\{e_1, \ldots, e_{m(n)}\}$; hence, $l_j^{(n)} \in \mathscr{L}_n$. We call

$$\Lambda_n \equiv \sup_{x\in K} \sum_{j=1}^{m(n)} |l_j^{(n)}(x)|$$

the nth *Lebesgue constant* for K, $A^{(n)}$. Given f defined on K, we let

$$(L_n f)(x) \equiv \sum_{j=1}^{m(n)} f(A_j^{(n)})\, l_j^{(n)}(x)$$

be the interpolating polynomial for f at the points $A^{(n)}$. Thus $L_n f \in \mathscr{L}_n$ and $(L_n f)(A_j^{(n)}) = f(A_j^{(n)})$. We say that the compact set K is *determining* for $\bigcup_n \mathscr{L}_n$ if whenever $h \in \bigcup_n \mathscr{L}_n$ satisfies $h = 0$ on K then $h \equiv 0$. For such sets, it is clear that there exists, for each n, a collection of $m(n)$ points $A^{(n)} \subset K$ with $V_n(A^{(n)}) \neq 0$.

7.2. THEOREM. *Let $K \subset \mathbf{R}^N$ be a compact set which is determining for $\bigcup_n \mathscr{L}_n$. For each n, let $A^{(n)}$ be a set of $m(n)$ points in K satisfying $V_n(A^{(n)}) \neq 0$. Given f bounded on K, if $\limsup_{n \to \infty} \Lambda_n^{1/n} = 1$ then $\limsup_{n \to \infty} \|f - L_n f\|_K^{1/n} = \limsup_{n \to \infty} d_n(f, K)^{1/n}$.*

Proof. Let $\varepsilon > 0$ be arbitrary. For each nonnegative integer n we may select a polynomial $p_n \in \mathscr{L}_n$ with

$$\|f - p_n\|_K^{1/n} \leqslant d_n(f, K)^{1/n} + \varepsilon.$$

Since $p_n \in \mathscr{L}_n$, we clearly have $L_n p_n = p_n$; hence

$$\begin{aligned}\|f - L_n f\|_K &= \|f - p_n + L_n p_n - L_n f\|_K \\ &\leqslant \|f - p_n\|_K + \Lambda_n \|p_n - f\|_K \\ &= (1 + \Lambda_n) \|f - p_n\|_K.\end{aligned}$$

From the assumption that $\limsup_{n \to \infty} \Lambda_n^{1/n} = 1$, and the fact that $\varepsilon > 0$ was arbitrary, we obtain the theorem.

Finally, we remark that arrays satisfying $\limsup_{n \to \infty} \Lambda_n^{1/n} = 1$ do in fact exist; for example, we can take, for each n, $A^{(n)} \subset K$ satisfying $\max_{x^{(n)} \subset K} |V_n(x^{(n)})| = |V_n(A^{(n)})|$; we refer to $A^{(n)}$ as a set of *n-Fekete points* for K. In this case, by definition, $\Lambda_n \leqslant m(n)$ and the sequence of numbers $\{m(n)\}$ satisfies $\lim_{n \to \infty} m(n)^{1/n} = 1$.

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